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DUALITY, CROSSING AND MacLANE'S COHERENCE

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ABSTRACT

It is shown that MacLane's rectangle, pentagon and hexagon identities in category theory, when applied in particle physics to duality diagrams or to rational conformal field theories in two dimensions, yield the necessary physical algebraic constraints.

§1. Introduction

In a recent series of papers [1–5] on rational conformal field theories in two dimensions, certain constraints were introduced on matrices, the knowledge of which determines the theory. These constraints are associated with certain transformations of graphs. An examination of their form shows that they are similar to constraints introduced by MacLane [6] many years ago in category theory. In the present article we explain the nature of these constraints and their relation to MacLane's theory. This will then allow us to apply MacLane's results (in a recently extended form [16], [17]) to determine a generating set for these constraints. The content of this paper was obtained in the mathematical physics seminar at Tel Aviv University in June and the early part of July of 1988 and circulated in preprint form at that time. It was presented in a slightly revised form at the conference on Hopf algebras at the University of Beersheba in early January 1989.

We begin with a brief introduction. In the diagrammatic description of particle interactions one may regard a diagram such as

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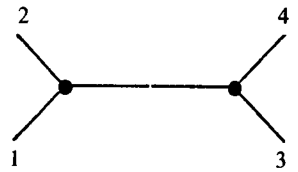
(1.1)



as indicating that particles 1 and 2 combine to form an intermediate particle which decays into particles 3 and 4.

"Crossing" refers to the assertion that there is a relation between the interaction described by the above diagram and the interactions associated to the diagrams

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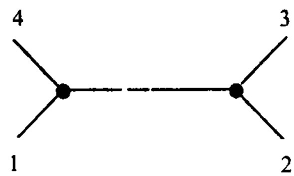
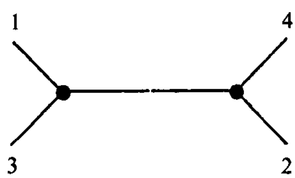


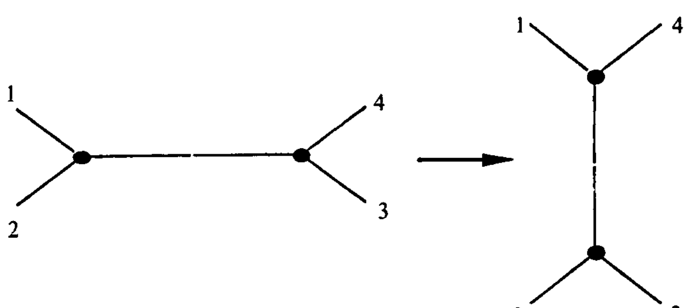
and

(1.3)

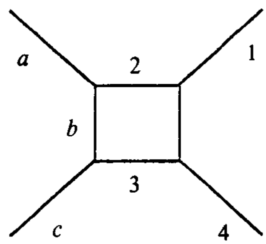


In its simplest form the assertion relates the analytic continuation of the amplitudes of these processes when expressed in terms of Lorentz invariant combinations of the four momenta of the interacting particles, the "Mandelstam variables" s, t, and u; cf. [7].

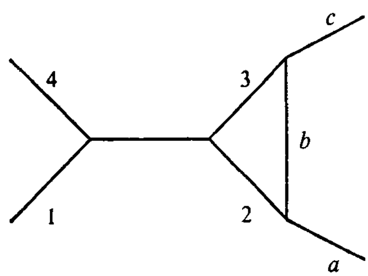
In the 1960's a systematic study of crossing in the context of "dual models" was begun [8]. With the advent of the Veneziano model [9] and its reinterpretation as a quantum string theory, duality was considered as a basic ingredient in the formulation of strong interaction physics [10]. In some of the early papers on dual models [11] one sees the "crossing transformation"



applied to an internal portion of a more complicated graph. Thus, for example, the above transformation, when applied to the graph



yields the graph

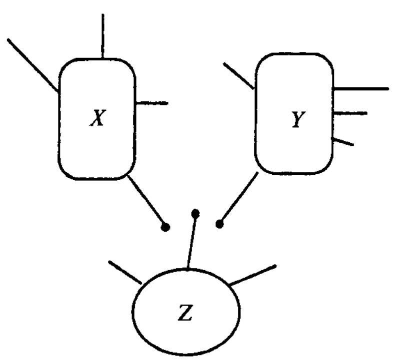


Recently [1–5], crossing transformations on graphs have reemerged in the context of finding constraints on two-dimensional rational conformal field theories. Roughly speaking, one associates to each crossing transformation a matrix which is, essentially, the matrix of a monodromy operator on a space of meromorphic sections of a line bundle over a Riemann surface whose knowledge determines the theory. Now one may pass from one graph to another by two different sequences of crossing transformations. Each of these routes will correspond to an expression in the basic matrices. Equating the expression corresponding to two different routes then provides a constraint on the basic matrices. It then becomes important to determine a set of generating relations from which all the others follow. In examining the nature of the generating relations one is struck by their similarity to the basic constraints introduced by MacLane in his celebrated paper [6] on tensor products in categories. Recently, MacLane's results have been extended by Joyal and Street to the case of "braided monoidal categories" [17]; see also [16]. The purpose of the present article is to explain the relation of the crossing constraints to MacLane's theory. (We gather from a footnote in [1] that Witten has also observed the similarity to MacLane's theory.) Moore and Seiberg [1] have presented a set of generating relations somewhat different from the ones presented here.

2. Constructions on graphs

In order to avoid confusion arising from differences in terminology between mathematicians and physicists we make the following definitions. A graph consists of two sets, E, its set of edges (or one-dimensional objects) and N its set of nodes (or zero-dimensional objects). Furthermore to each edge, e, is associated a subset $\{n_1, n_2\}$ of N called its boundary nodes. We say that n_1 and n_2 are incident to e. (We do not, at this stage, exclude the possibility that n_1 might equal n_2.) In the physics literature, nodes that are incident to more than one edge are called vertices. But in the mathematical terminology the word vertex refers to all nodes. So we will avoid the use of the word vertex altogether, and speak of internal and external nodes. We will consider graphs where every node is incident to either three edges (the internal nodes) or to one edge (the external nodes). In physics language this means that we are considering graphs similar to the Feynman diagrams occurring in a ϕ^3 theory. External edges (those attached to external nodes) will be called legs. We can consider two types of constructions:

Joining: We can pick one leg from each of three graphs and join them. So



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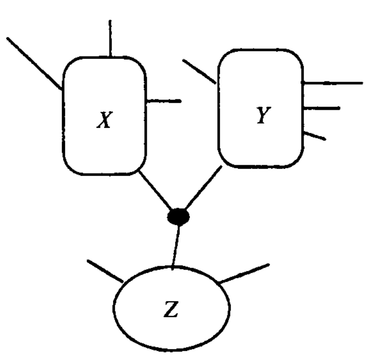
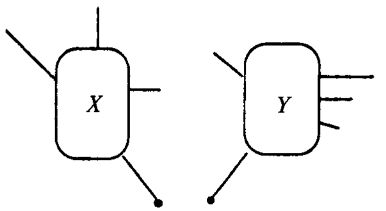


(In this figure X, Y and Z stand for various graphs whose internal structure we have not drawn.)

Or we can pick two legs from one graph and join it to one from another. So



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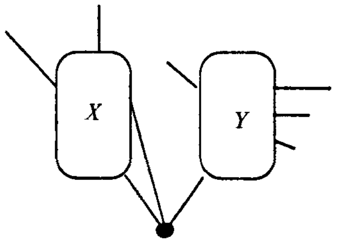
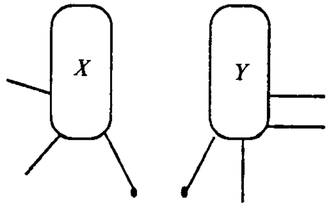


Or we can join three legs of a single graph.

Crossing: If (1.1) occurs as a subdiagram of a graph, then it is replaced by (1.2) or (1.3) (with all the other connections unchanged).

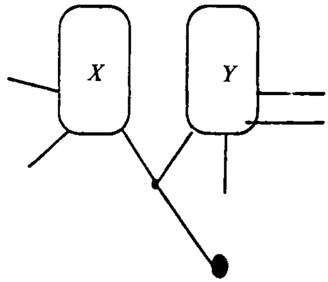
An illustration of this operation was given in the preceding section.

Suppose that we consider connected graphs with one leg with an exterior node marked. At the risk of mixing metaphors, we will call such a graph a rooted graph, and the marked leg the root. Then joining becomes a binary operation, J, on rooted graphs, by joining the two roots to the trivial graph, consisting of a single edge and two nodes, and then marking the remaining node. Thus



goes to

$\overset{J(X,Y)}{=}$



Notice that the operation J does not introduce any new loops (cycles). In particular, if X_1 and X_2 are trees (have no cycles), then $J(X_1, X_2)$ is again a tree. Indeed, let n_1 and n_2 denote the number of nodes in X_1 and X_2 and e_1 and e_2 the number of edges. Then

$$n = n_1 + n_2$$

and

$$e = e_1 + e_2 + 1$$

where n and e denote the number of nodes and edges in $J(X_1, X_2)$. So if $n_1 = e_1 + 1$ and $n_2 = e_2 + 1$, then $n = e + 1$.

3. Cyclically ordered graphs

For any finite set, A, let $S(A)$ denote the group of all one-to-one transformations of A onto itself. So if A has n elements, the group $S(A)$ is isomorphic to the symmetric group $S_n = S(\{1, 2, \ldots, n\})$.

Suppose that A has n elements. By a cyclic order on A we mean an element, s, of $S(A)$ which has order n. There are thus $(n-1)!$ different cyclic orders on A. By a cyclically ordered graph we shall mean a graph for which a cyclic order has been chosen at each (internal) node.

When we perform the operation J on two cyclically ordered graphs, X and Y, we must decide which of the two cyclic orders to put on the joining node. If x denotes the root (the marked leg) of X and y denotes the root of Y and j denotes the new leg, we must choose between the cyclic orders (j, x, y) and (j, y, x) at the new node. We will denote the first choice by $J(X, Y)$ and the second choice by $J(Y, X)$. We have thus proved

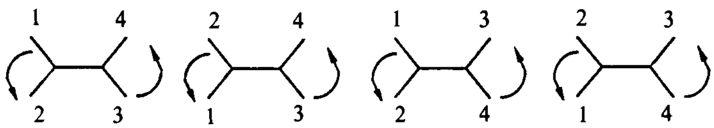
PROPOSITION. *The join operation defines a nonsymmetric binary operation, J, on cyclically ordered graphs. If X and Y are trees, then so is $J(X, Y)$.*

Let us consider the basic four leg graph

and let $G = G(\succ\!\!-\!\!\prec)$ denote $S(\mathrm{Legs}(\succ\!\!-\!\!\prec))$, so G is isomorphic to S_4 if we label the legs as in (1.1).

There are three unordered graphs corresponding to (1.1), (1.2) and (1.3) and the group G acts transitively on this three-element set (the action of S_4 on unordered partitions of $\{1, 2, 3, 4\}$ into two unordered subsets).

Each of these unordered graphs gives rise to four cyclically ordered graphs. Thus



A *B* *C* *D*

are the four cyclically ordered graphs corresponding to (1.1).

The group G acts transitively on the twelve-element set of all cyclically ordered graphs. The element (13)(24) fixes the graph A above.

(If we were to consider oriented cyclically ordered graphs, where an orientation is chosen on each edge, the (13)(24) would not act trivially and the orbit of G would contain twenty-four distinct possibilities.)

If $\succ\!\!-\!\!\prec$ is an internal subgraph of some unordered graph X_0, then we have the two graphs Y_0 and Z_0 associated to X_0 by crossing, and G acts transitively on this three-element set as before. Let us choose a cyclic order on X_0. We denote the corresponding cyclically ordered graph by X. Conversely, if X is cyclically ordered, we let X_0 denote the corresponding unordered graph. Suppose that X is a cyclically ordered graph, $\succ\!\!-\!\!\prec$ is a four-legged subdiagram of X and $b \in G(\succ\!\!-\!\!\prec)$. We let bX be the cyclically ordered graph given by cyclically ordering the nodes of bX_0 as follows: each node of bX_0 other than the two nodes corresponding to the nodes of $\succ\!\!-\!\!\prec$ comes from a unique node of X_0. On each such node we put the old cyclic ordering as in X. At the nodes corresponding to those of the crossing subgraph, $\succ\!\!-\!\!\prec$, we put the cyclic ordering given by the action of G as described above.

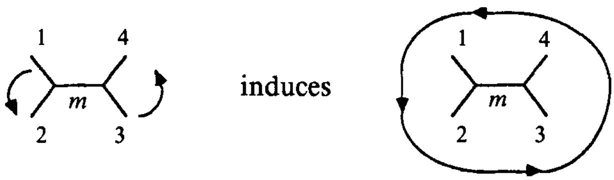
We can now describe "moves" on cyclically ordered graphs:

Move: For a cyclically ordered graph, X, pick a four-legged subgraph, $\succ\!\!-\!\!\prec$.

Then pick an element $b \in G(\succ\!\!-\!\!\prec)$ and obtain the cyclically oriented graph bX by the procedure given above.

The mathematical problem that then arises is: describe the relations among the moves. That is, describe those sequences of moves that will lead from X back to the same graph X. We must be precise about what we mean by "the same X": Notice that the various legs of the graph retain their identity at each move. So after a sequence of k moves, we obtain a graph Y together with an identification of the legs of X with the legs of Y. We say that "Y is the same as X" if there is a one-to-one map (of the edges and modes) of X onto (the edges and nodes of) Y which preserves all incidences and cyclic orders, and which reduces to the given identification on the legs.

We can also consider a more restricted class of moves defined as follows: Suppose we consider a four-legged graph $\succ\!\!-\!\!\prec$. We claim that a cyclic order on the nodes of $\succ\!\!-\!\!\prec$ induces a cyclic order on the legs of $\succ\!\!-\!\!\prec$:



In other words, the cyclic orders $(12m)$ and $(m34)$ induce the cycle (1234). (Notice that if we choose the nodes in the opposite order we get the same cycle, $(3412) = (1234)$.)

Notice that (1234) carries

A $(1234)A$

and the cycle associated to $(1234)A$ is again (1234). Notice also that

$$(1234)^2 = (13)(24)$$

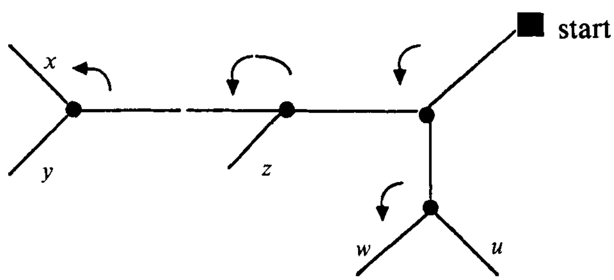
which carries A into itself, as is to be expected.

Restricted moves: Let X be any cyclically ordered graph. Each four-legged subgraph, $\succ\!\!-\!\!\prec$, is cyclically ordered, and hence determines an element, a, of $G(\succ\!\!-\!\!\prec)$. Apply this preferred element a to X so as to move to aX. Then choose another four-legged subgraph etc. These are the restricted moves.

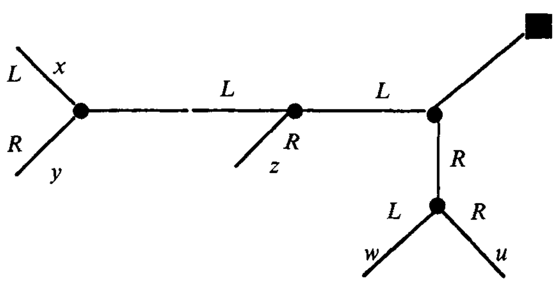
4. Restricted moves on rooted trees

In this section we will show that a cyclically ordered rooted tree is a scheme for forming tensor products. (In more technical language it is a binary word in the sense of [12, p. 161] but where we don't allow for the empty word.) We shall also see that a restricted move on a rooted tree is just an associativity operator in the sense of MacLane, and hence we can apply Theorem 3.1 of [6] to determine all the relations among restricted moves on cyclically ordered rooted trees.

Suppose we are given a rooted tree. We can regard the cyclic order on the nodes as a "sorting procedure" on the remaining legs as follows: Start from the root (the marked leg). At its juncture, mark the first edge encountered in the cyclic order by L and the second edge by R. This is to be interpreted as follows: All legs connected to the edge marked L will be placed to the left of all the legs attached to the edge marked R. If the edge we just marked L is not a leg, continue to its other joining node and repeat the procedure of marking the first edge encountered in the cyclic order by L and the second by R. Do the same for the edge marked R if it is not a leg. Continue. Now consider each interior node as a tensor product and hence each interior edge as a parenthesis system. Thus, for example,



is marked as



and corresponds to the "tensor product"expression

$$((x \otimes y) \otimes z) \otimes (w \otimes u).$$

Here is another way of saying the same thing (in the language of [12, p. 161]). Any cyclically ordered rooted tree can be built from smaller trees by the operation J unless it is already the trivial tree consisting of a single edge. So we let this trivial tree be denoted by __ and write

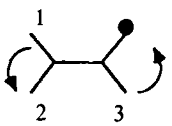
$$X \otimes Y \quad \text{instead of} \quad J(X, Y).$$

Thus we have proved:

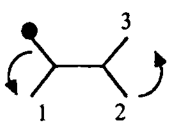
THEOREM. *There is a bijective correspondence between cyclically ordered rooted trees and binary words. The correspondence can be established by representing the tree by successive application of the join operation.*

In short: every cyclically ordered rooted tree is a binary word in the sense of [12, p. 161] with no e_0.

Now consider the four-legged graph



where we have marked the leg 4. It corresponds to $(1 \otimes 2) \otimes 3$. The operator $a = (1234)$ applied to this graph gives



which corresponds to $1 \otimes (2 \otimes 3)$. Thus a is the "formal associativity operator".

If $\succ\!\!-\!\!\prec$ is an interior subgraph of a rooted tree, X, one and only one of its legs will lead to the root, since X is a tree, and we may consider this leg of $\succ\!\!-\!\!\prec$ as marked. Hence every restricted move on a cyclically ordered rooted tree is an *instance* of an associativity in the terminology of [6].

Now we can read the relations on restricted moves on rooted trees directly from MacLane's paper. They are generated by relations of two types:

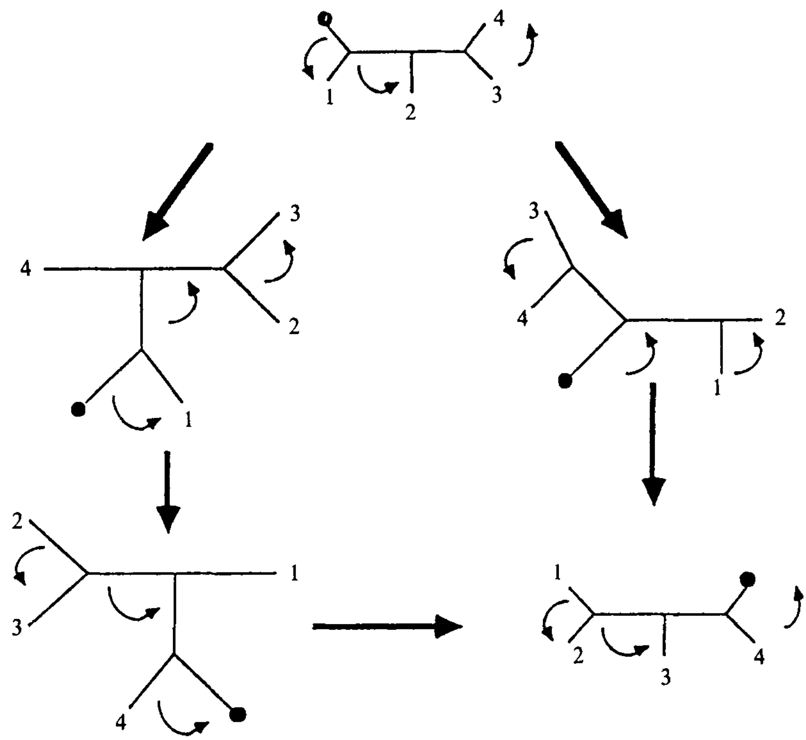
(1) A "rectangle identity" or "naturality" which asserts that a restricted move applied to a subtree attached to one of the unmarked legs of $\succ\!\!-\!\!\prec$ "commutes" with the restricted move associated to the four legged subdiagram $\succ\!\!-\!\!\prec$. Put another way, the moves associated to disjoint subdiagrams commute.

(2) A "pentagon identity" which relates the two paths going from

$$1\otimes(2\otimes(3\otimes 4)) \quad \text{to} \quad ((1\otimes 2)\otimes 3)\otimes 4$$

one in two moves and the other in three moves.

In our present presentation we would draw MacLane's pentagon identity as



See, in relation to the pentagon identity, the papers [1] and [4].

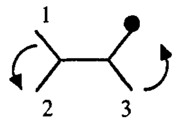
MacLane was interested in the following application: suppose we postulate a tensor product on a category, C. That is a functor from $C\times C$ to C denoted by $\otimes$. An associativity isomorphism is then a natural transformation

$$a = a(A, B, C): A\otimes(B\otimes C)\leftarrow(A\otimes B)\otimes C$$

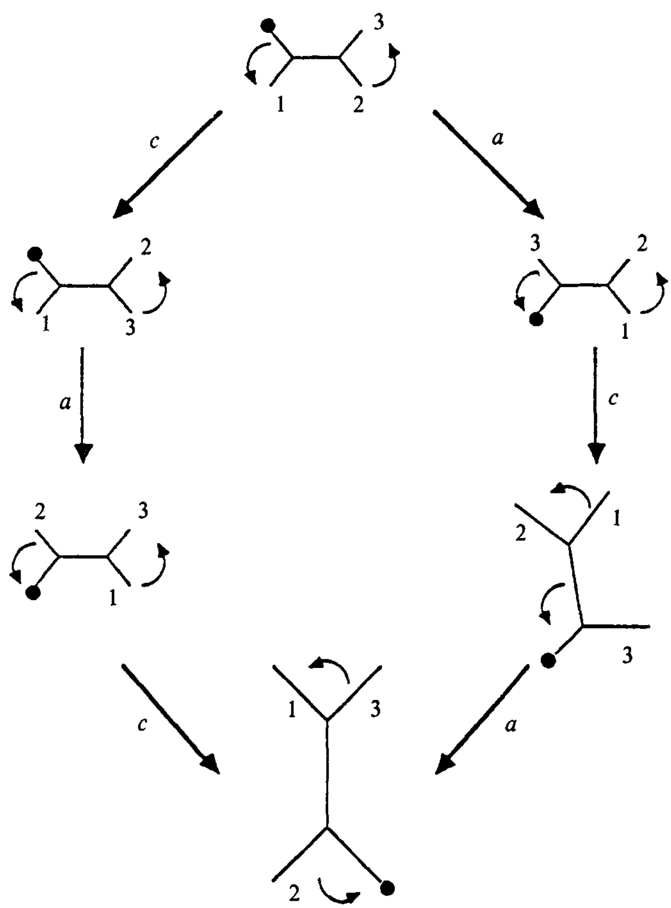
such that $a(A, B, C)$ has a two-sided inverse. The question was: what are the conditions on these a's so that there is a unique natural transformation between any two n-fold tensor products? His answer, Theorem 3.1 of [6], is that the pentagon identity (together with the naturality) is necessary and sufficient.

We shall describe a different application of MacLane's result in the next section.

A four cycle and any transposition (two cycle) generate S_4. On any rooted tree each four-legged subdiagram, $\succ\!\!-\!\!\prec$, has a marked edge as we have observed, and hence a unique transposition in $G(\succ\!\!-\!\!\prec)$, the transposition of the two legs not joined to the marked edge. This transposition clearly corresponds to an instance of commutativity transformation in the tensor product interpretation. Indeed the diagram



corresponds to $(1 \otimes 2) \otimes 3$, and thus interchanging 1 and 2 is a commutativity operator. Hence we can read from MacLane's paper the hexagon identity relating associativity and commutativity. In terms of our four-legged graphs, MacLane's hexagon identity is



Theorem 5.1 of [6] asserts that the hexagon identity together with the pentagon identity and naturality determine all the relations on moves on cyclically ordered rooted trees. In particular the braid relations follow as MacLane proves in [6].

5. Moves on labelled graphs

Let the group G act on the set P. Let $F(P, L)$ denote the set of all functions from P to L (i.e. the set of all "labels" of elements of P by elements of L). Then G acts on $F(P, L)$ by the rule

$$(bf)(p) = f(b^{-1}p).$$

We need another notion. Let K be a finite set. A vector bundle $E \to K$ is a rule which associates a vector space E_k to each $k \in K$. Suppose we are given an action of G on K. Then an action of G on E (consistent with the given action on K) is a rule which associates to each $b \in G$ and to each $f \in K$ a linear transformation

$$b : E_f \to E_{bf}$$

so that we get an action of G on E. Thus the identity of G gives the identity transformation for each E_f and

$$E_f \to E_{bf} \to E_{cbf} \quad \text{equals} \quad E_f \to E_{cbf}.$$

Suppose that all the E_f have the same dimension and have been identified with the fixed vector space, V. Then $b : E_f \to E_{bf}$ can be identified with a linear transformation

$$A(f, b) : V \to V.$$

The consistency condition that says we have a group action then becomes

$$A(bf, c)A(f, b) = A(f, cb). \tag{5.1}$$

If V has a preferred basis the A's become matrices and (5.1) becomes a matrix equation.

Now consider a cyclically ordered graph X in which we have labelled all the edges by labels from a finite set, L. Let $\succ\!\!-\!\!\prec$ be a four legged subdiagram of X and $b \in G(\succ\!\!-\!\!\prec)$. Then each edge, e, of the graph bX other than the new central connecting edge corresponds to an edge of X, and hence carries a label. Thus we label all these edges according to our standard rule,

$$(bh)(e) = h(b^{-1}e).$$

For the central edge we proceed as follows: We consider the vector bundle over the space of labels on the legs of $\succ\!\!-\!\!\prec$ whose fiber is C^L.

We assume that we are given an action of G on this vector bundle. In other words, to each $b \in G(\succ\!\!-\!\!\prec)$ and to each label f on the legs of $\succ\!\!-\!\!\prec$ we are given a matrix

$$A(f, b)_{qp}, \qquad p, q \in L$$

satisfying (5.1).

Let $(bX)_q$ denote the graph bX with the label q on the new central edge and with all the other edges labelled as above. Then we move X into

$$\Sigma_q A(f, b)_{qp}(bX)_q.$$

We have moved a labelled graph X into a linear combination of labelled graphs. Thus our moves are on the space of linear combinations of labelled graphs.

Notice that after a finite number of steps the sum will be over various labels, but not over labels of legs. So we once again obtain consistency relations, since we get from one labelled graph to a sum of others with the same external labels by several routes, and the two sums over the internal labels must be equal. This imposes a set of constraints on the matrices A, above and beyond (5.1). Similarly, we may consider the constraints arising from restricted moves. In either event the generating relations were described in the preceding section.

6. Applications to rational conformal field theories

Rational conformal field theories are characterized by (i) the central charge c of the left and right Virasoro algebras (in mathematical language, the choice of an element in $H^2(g)$ where g is the Lie algebra of Fourier polynomial vector fields on the circle), (ii) the finite set of primary fields and their conformal weights (h_i, h_i), and (iii) by the structure constants C_{IJK} of the operator product algebra (O.P.A.) [13].

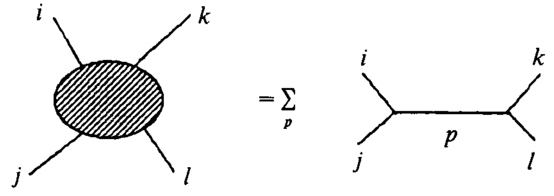
Factorization of the n-point function and the structure of the conformal families leads to the construction of conformal blocks [13]. (For simplicity we consider minimal theories only. The extension to other cases is not hard. We do not distinguish between fields and their conjugates to avoid complicated notation.) For example, the 4-point function can be written as

(6.1)
$$\begin{aligned} &\langle \phi_i(z_1, \bar{z}_1)\phi_j(z_2, \bar{z}_2)\phi_k(z_3, \bar{z}_3)\phi_l(z_4, \bar{z}_4)\rangle \\ &\sim \sum_p c_{ijp}c_{pkl} I \begin{pmatrix} j & k \\ i & l \end{pmatrix}_p (z) \times \text{“c.c.”} \end{aligned}$$

where by "c.c." we mean the corresponding expression with the bar variables and z is the cross ratio

$$z = \frac{z_1 - z_2}{z_1 - z_3} \frac{z_4 - z_2}{z_4 - z_3}.$$

The I's are the holomorphic conformal blocks of the 4-point function. The blocks span a vector space of meromorphic functions. Equation (6.1) is described pictorially by

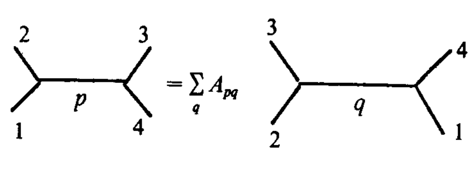


We can now make contact with the formalism introduced in the previous sections. We see that we can associate to each holomorphic (antiholomorphic) conformal block a 4-leg labelled graph of the kind discussed previously. Similarly we can associate a labelled graph to the holomorphic blocks of the n-point function. The legs of the graph usually correspond to some primary fields while the internal edges correspond to summation over descendants of a given conformal family.

Crossing: The completeness of the s- and t-channel blocks of the 4-point function leads to the relation [14, 1, 5]:

$$c_{12p}c_{p34}I\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}_p (z) = \sum_q A\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}_{pq} c_{23q}c_{q14}I\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}_q (1-z) \tag{6.2}$$

which can be described graphically by



The matrices A are therefore the matrix representation of the associativity operator a. (The matrices A are called F in [1].)

The commutativity matrices are obtained by the same reasoning. They are diagonal matrices whose entries depend on the h_i's.

For example, the matrix of basic commutativity operator discussed in Section 4, $(1\otimes 2)\otimes 3 \to (2\otimes 1)\otimes 3$, is given by

$$C\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}_{pq} = e^{i\pi(h_p - h_1 - h_2)}\delta_{pq}. \tag{6.3a}$$

whereas the matrix corresponding to $(1\otimes 2)\otimes 3 \to 3\otimes(1\otimes 2)$ is given by

$$\tilde{C}\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}_{pq} = e^{i\pi(h_4 - h_p - h_3)}\delta_{pq}. \tag{6.3b}$$

(The matrices C are called Ω in [3].)

The matrices A and C obey the consistency condition (5.1), in particular they act on the labels of the graphs as elements of S_4. A special case is the braiding matrix B; according to (5.1) it is given by $B = \tilde{C}AC$. (The matrices B are called R in [3].)

However C^2 may not be equal to the identity, since it follows from (6.3) that C^2 is a scalar matrix consisting of a possibly non-trivial phase factor. (We wish to thank Prof. MacLane for pointing out an error in our original treatment at this point.) Hence the original MacLane coherence theorem involving commutativity does not directly apply. But a more recent generalization of MacLane's theorem does apply. A *braided monoidal category*, cf. [16] and [17], is a category with a tensor product defined axiomatically together with an associativity, a, and a commutativity, c, but where c^2 is not assumed to be the identity. One assumes the pentagon identity and two hexagon identities — the hexagon identity of Section 4 for c and the same identity for c^{-1}. (Of course these reduce to one and the same identity if $c^2 = 1$.) The coherence theorem of Joyal and Street [17] then asserts that a diagram build up from instances of a and c using tensor products and composition commutes if and only if the associated braids are equal. (We wish to thank Prof. MacLane for referring us to the papers [16] and [17] and to the use of the Joyal–Street coherence theorem at this juncture.)

We have to justify the fact that the same matrices A and C represent the associativity and commutativity operators inside higher-order graphs. For this we need to know that conformal blocks with descendants as external legs transform with the same A and C matrices. In addition we need to know that we can isolate a 4-leg graph inside a higher order and operate on it with operations defined on conformal blocks of the 4-point function. The latter point can be proved by using the operator product expansion in the n-point function and by summing and desumming over descendants [15]. The first point can be easily proved in the case that there is no mixing between the blocks, i.e. no integer differences between any of the h_i's. This is done by using the linear relation between blocks with external descendants and external primaries [13]. In the case that mixing does occur the blocks do not form an orthogonal basis to begin with [14]. In many cases there exists an extended algebra which allows the separation of the mixed blocks.

With this in mind the identification of operations on holomorphic blocks with our general formalism is complete. The demand of coherence is equivalent to the demand that the n-point function be well defined and compatible with the O.P.A. The matrices A and C therefore obey MacLane's pentagon and two hexagon identities, one involving C and the other involving C^{-1}.

The equations resulting from the pentagon identity are

$$\sum_p A\begin{pmatrix} 2 & 3 \\ r & 4 \end{pmatrix}_{sp} A\begin{pmatrix} 1 & p \\ 5 & 4 \end{pmatrix}_{rq} A\begin{pmatrix} 1 & 2 \\ q & 3 \end{pmatrix}_{pt} = A\begin{pmatrix} 1 & 2 \\ 5 & s \end{pmatrix}_{rt} A\begin{pmatrix} t & 3 \\ 5 & 4 \end{pmatrix}_{sq}. \tag{6.4}$$

They can be obtained from the pentagon identity in Section 4 in the following way: The marked leg is labelled 5. The internal edges of the top 5-legged graph are labelled r and s. Equation (6.4) is obtained by equating the coefficients of the conformal block of the 5-point function with internal indices q and t for each q and t separately.

Similarly, the equations resulting from the hexagon for a and c are obtained by equating the coefficients of the block

$$I\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_r$$

of the 4-point function. The marked leg in the hexagon diagram in Section 4 is labelled 4 and the internal edge of the top 4-legged graph is labelled p. Then the hexagon identity yields

$$\sum_q A\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}_{pq} \tilde{C}\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}_{qq} A\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}_{qr} = C\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}_{pp} A\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}_{pr} C\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}_{rr} \tag{6.5}$$

and the corresponding equation with C^{-1}.

Due to the coherence theorem, these equations in addition to the defining relations (5.1) are the complete set of equations obeyed by the matrices A and C. We have checked that the Ising model (which is the simplest case with non-trivial A's) is a solution of the hexagon and pentagon equations.

7. Concluding remarks

(1) We have indicated the applications of the results of Sections 2–5 to conformal field theories in two dimensions. But it seems to us that there might be other areas of applications of the types of problems discussed, in particular to string theories and to statistical mechanics.

(2) There are some immediate mathematical problems that we have not dealt with here but hope to study in the future. The most pressing, of course, is to find the structure of the solutions to the matix equations described in Section 5. In addition, one would want to extend the results of Section 4 to the case where loops are allowed, so that MacLane's theorem does not immediately apply. Also, one might want to consider more general types of graphs, not necessarily those with three edges incident to every internal node.

(3) It would be of interest to explore the relations of the problems studied here to other areas of mathematics, for example, the relation to knot theory and the "category of tangles"; cf. [18] and the references given there. Also the relation to the work on quantum groups as developed in the Leningrad school. In fact, it was a lecture by Prof. Faddeev at the Landau conference at Tel Aviv which stimulated the interest of one of us (YN) in the current topic. It would also be interesting to try to understand if there is a "continuous analogue" of the constructions in Sections 4 and 5, where "crossing" is replaced by some form of surgery or cobordism. For example, the idea of replacing the deleted edge in the crossing operation by a vector bundle is highly reminiscent of a blowing up operation as it appears in birational equivalence; cf. [19], for example.

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